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U(N) spinning particles and higher spin equations on complex manifolds

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Abstract: Guided by a spinning particle model with U(N)-extended supergravity on the worldline we derive higher spin equations on complex manifolds. Their minimal formulation is in term of gauge fields which satisfy suitable constraints. The latter can be relaxed by introducing compensator fields. There is an obstruction to define these systems on arbitrarily curved spaces, just as in the usual theory of higher spin fields, but we show how to couple them to Kähler manifolds of constant holomorphic curvature. Quite interestingly, the first class gauge algebra defining the U(N) particles on these manifolds is quadratic and realizes the zero mode sector of certain nonlinear U(N) superconformal algebras introduced sometimes ago by Bershadsky and Knizhnik in 2D.

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Contents

1 Introduction

Spinning particle models are quite useful for describing field theories in first quantization. In particular, spinning particles with gauged $O(N)$-extended susy on the worldline [1, 2] can be used to describe properties of fields of spin $N/2$ [3–5].

In this paper we analyze spinning particles with gauged U(N)-extended susy on the worldline and use them to derive gauge invariant higher spin equations on certain complex manifolds. The U(N) particles for $N = 1, 2$ were originally introduced in [6] as a dimensional reduction of the $N = 2$ string, and generalized to arbitrary N in [7]. Their Dirac quantization introduces constraints on the particle Hilbert space that are interpreted as equations of motion for certain tensor fields with holomorphic indices satisfying the symmetries of a rectangular Young tableau [7]. We analyze these equations on the flat complex space $\mathbb{C}^d$. By integrating a subset of them in terms of gauge potentials we are led to gauge invariant field equations which are quite similar in form to the higher spin equations introduced by Fronsdal [8].

An example of these equations is that of a gauge field $\varphi_{\mu_1\ldots\mu_N}$ with N symmetric holomorphic indices (we use complex coordinates $x^\mu, \bar{x}^{\bar{\mu}}$ of $\mathbb{C}^d$; tensor indices are raised and lowered with the flat hermitian metric $\delta_{\mu\bar{\nu}}$). It satisfies the equation

$$\partial_\alpha \bar{\partial}^\alpha \varphi_{\mu_1\ldots\mu_N} - \sum_{i=1}^{N} \partial_{\mu_i} \bar{\partial}^\alpha \varphi_{\mu_1\ldots\alpha\ldots\mu_N} = 0 \tag{1.1}$$

where the α index in the second term is located in i-th position. The gauge invariance is given by

$$\delta\varphi_{\mu_1\ldots\mu_N} = \partial_{\mu_1}\lambda_{\mu_2\ldots\mu_N} + \text{ cyclic perm.} \tag{1.2}$$

where the gauge parameter $\lambda_{\mu_2 \ldots \mu_N}$ has $N-1$ symmetric holomorphic indices and is constrained by $\bar\partial^\alpha \lambda_{\alpha\mu_3\ldots\mu_N} = 0$. For consistency the gauge field must also satisfy a differential constraint $\bar\partial^\alpha \bar\partial^\beta \varphi_{\alpha\beta\mu_3\ldots\mu_N} = 0$. These equations are very much reminiscent of Fronsdal's equations. Since there is no invariant concept of taking traces on holomorphic indices, the usual trace constraints that appear in Fronsdal's formulation are naturally substituted here by differential constraints.

The constraints on gauge fields and on gauge parameters can be relaxed by adding compensator fields. For example in the above case with $N = 2$, one can introduce a single compensator field ρ and the equation reads

$$\partial_\alpha \bar\partial^\alpha \varphi_{\mu\nu} - \partial_\mu \bar\partial^\alpha \varphi_{\alpha\nu} - \partial_\nu \bar\partial^\alpha \varphi_{\mu\alpha} = \partial_\mu \partial_\nu \rho \tag{1.3}$$

with gauge symmetry

$$\delta\varphi_{\mu\nu} = \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu \,, \qquad \delta\rho = -2\bar\partial^\alpha \lambda_\alpha \,. \tag{1.4}$$

This is reminiscent of the Francia-Sagnotti construction [9] for relaxing the trace constraints of standard higher spin gauge theories using compensator fields.

We derive equations also for more general tensor fields with the symmetry type of a rectangular Young tableaux with p rows and N columns. We do so by using the compact notation provided by the quantum mechanical operators of the spinning particle.

The equations just discussed are defined on a flat complex space, viewed as a Kähler manifold, but it is interesting to study if they can be extended to more general Kähler spaces. While it is known that the U(N) particles for $N = 1, 2$ can be coupled to any Kähler manifold [6], it was thought that for $N > 2$ these particles could only be consistent on flat manifolds, as the standard susy transformation rules do not leave the particle action invariant on a curved space [7]. We can actually show that a coupling is still possible for Kähler manifolds with constant holomorphic curvature. To achieve this result we use a hamiltonian approach and notice that the algebra of first class constraints defining the model closes on Kähler manifolds with constant holomorphic curvature, though in a nonlinear way. In fact we obtain a quadratic first class algebra that, quite interestingly, is seen to coincide with the zero mode sector of two dimensional nonlinear U(N) superconformal algebras, introduced sometimes ago by Bershadsky and Knizhnik [10, 11]. This result is consistent with [7] in that the susy transformations rules associated to a nonlinear algebra differ from the one employed in [7]. The corresponding gauge invariant differential equations can similarly be defined on such complex spaces.

Having understood that U(N) particles and related gauge invariant field equations can be defined on a non trivial class of curved spaces, it is interesting to study their quantum properties. We begin this analysis using a first quantized path integral description. This worldline approach is quite flexible and efficient, and by using closed worldlines one can study directly the one loop effective action associated to the field equations described above. To construct the path integral is necessary to gauge-fix the particle action and identify the correct measure over the moduli space of inequivalent gauge choices. We start considering a flat target space and compute the physical degrees of freedom. This gives a check on the path integral measure, which can be used to compute more general observables.

While the complex nature of target space does not suggest us an immediate physical application of these higher spin equations (either the target space has an even number of times, or no time direction at all) they might still be useful to describe properties of complex manifolds or for developing additional intuition on the standard theory of higher spin fields (see [12] for reviews). From this point of view it would be quite interesting to search for consistent nonlinear extensions of the free equations described here.

We structure our paper as follows. We review the U(N) spinning particle in section 2 using a hamiltonian formulation. Dirac quantization is analyzed in section 3. In particular we describe in this section how the constraints ("curvature formulation") can be partially integrated to produce the equations of motion for gauge fields introduced above ("gauge field formulation"). In section 4 we prove the consistency of the coupling to Kähler manifolds of constant holomorphic curvature. In section 5 we construct the worldline path integral and compute the number of physical degrees of freedom. Section 6 contains our conclusions and an outlook.

2 The U(N) spinning particle in flat space

We consider an even dimensional flat space, viewed as the flat Kähler manifold $\mathbb{C}^d$, with $D = 2d$ real dimensions; the bosonic fields $x^M(\tau)$, interpreted as space-time coordinates, split into complex components $x^\mu(\tau)$ and $\bar{x}^{\bar{\mu}}(\tau)$, with $\mu = 1 \dots d$. They are paired with fermionic superpartners $\psi_i^\mu(\tau)$ and $\bar{\psi}^{\bar{\mu}i}(\tau)$, $i = 1 \dots N$, belonging to the $\mathbf{N}$ and $\bar{\mathbf{N}}$ of U(N), respectively. The flat metric in complex coordinates is simply $\delta_{\mu\bar{\nu}}$, the other components being zero. With these ingredients the phase space action

$$S = \int_0^1 d\tau \left[p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}^i_\mu \dot{\psi}^\mu_i - p_\mu \bar{p}^\mu \right] \tag{2.1}$$

describes the motion of a free particle with a pseudoclassical spin associated to the Grassmann coordinates. This system enjoys various conserved quantities, including those corresponding to the U(N)-extended supersymmetry on the worldline

$$H = p_\mu \bar{p}^\mu\,, \quad Q_i = \psi_i^\mu p_\mu\,, \quad \bar{Q}^i = \bar{\psi}^{\bar{\mu}i} \bar{p}_{\bar{\mu}}\,, \quad J_i^{\ j} = \psi_i^\mu \bar{\psi}^j_\mu \tag{2.2}$$

where indices are lowered and raised using the $\delta_{\mu\bar{\nu}}$ metric and its inverse. We have chosen normalizations so that H is real, $(Q_i)^* = \bar{Q}^i$, and $(J_i^{\ j})^* = J_j^{\ i}$, so that $J_i^{\ i}$ is real for any fixed i. The fundamental Poisson brackets are easily read off from the symplectic term of the action

$$\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu\,, \quad \{\bar{x}^{\bar{\mu}}, \bar{p}_{\bar{\nu}}\}_{PB} = \delta^{\bar{\mu}}_{\bar{\nu}}\,, \quad \{\psi_i^\mu, \bar{\psi}^{\bar{\nu}j}\}_{PB} = -i\delta^{\mu\bar{\nu}} \delta_i^j \tag{2.3}$$

and the above conserved charges generate symmetry transformations through Poisson brackets (using $\delta z = \{z, \mathcal{G}\}_{PB}$ with $\mathcal{G} \equiv \xi H + i\bar{\epsilon}^i Q_i + i\epsilon_i \bar{Q}^i + \alpha_i^{\ j} J_j^{\ i}$)

$$\begin{aligned} \delta x^\mu &= \xi \bar{p}^\mu + i\bar{\epsilon}^i \psi_i^\mu\,, & \delta \bar{x}^{\bar{\mu}} &= \xi p^{\bar{\mu}} + i\epsilon_i \bar{\psi}^{\bar{\mu}i} \\ \delta \psi_i^\mu &= -\epsilon_i \bar{p}^\mu + i\alpha_i^{\ j} \psi_j^\mu\,, & \delta \bar{\psi}^{\bar{\mu}i} &= -\bar{\epsilon}^i p^{\bar{\mu}} - i\alpha_j^{\ i} \bar{\psi}^{\bar{\mu}j} \\ \delta p_\mu &= 0\,, & \delta \bar{p}_{\bar{\mu}} &= 0\,, \end{aligned} \tag{2.4}$$

which correspond to rigid time translations with parameter ξ, N complex supersymmetries with grassmannian parameters ϵ_i and $\bar{\epsilon}^i$, and U(N) rotations parametrized by α^i_j. The explicit U(N)-extended supersymmetry algebra is easily computed

$$
\begin{aligned}
&\{Q_i, \bar{Q}^j\}_{PB} = -i\delta_i^j H \\
&\{J_i^{\,j}, Q_k\}_{PB} = -i\delta_k^j Q_i\,, \qquad \{J_i^{\,j}, \bar{Q}^k\}_{PB} = i\delta_i^k \bar{Q}^j \\
&\{J_i^{\,j}, J_k^{\,l}\}_{PB} = i\delta_i^l J_k^{\,j} - i\delta_k^j J_i^{\,l}
\end{aligned} \tag{2.5}
$$

with other independent Poisson brackets vanishing.

The model we are interested in is obtained by gauging this algebra of first class constraints through the introduction of corresponding gauge fields: an einbein $e(\tau)$ for time translations, complex gravitini $\chi_i(\tau)$ and $\bar{\chi}^i(\tau)$ for the extended supersymmetry, and a U(N) gauge field $a^i_j(\tau)$ for the rotations. These fields correspond to the gauge fields of a U(N)-extended supergravity on the worldline, and the full action of the U(N) spinning particle becomes

$$
S = \int_0^1 d\tau \big[p_\mu \dot{x}^\mu + \bar{p}_{\bar{\mu}} \dot{\bar{x}}^{\bar{\mu}} + i\bar{\psi}^i_\mu \dot{\psi}^\mu_i - e\,\underbrace{p_\mu \bar{p}^\mu}_{H} - i\bar{\chi}^i \underbrace{p_\mu \psi^\mu_i}_{Q_i} - i\chi_i \underbrace{\bar{p}_{\bar{\mu}} \bar{\psi}^{\bar{\mu}\, i}}_{\bar{Q}^i} - a^i_j (\underbrace{\psi^\mu_i \bar{\psi}^j_\mu}_{J_i^{\,j}} - s\delta^j_i)\big] \tag{2.6}
$$

where we have inserted also a Chern-Simons coupling s for the U(1) part of the gauge group U(N), since it is invariant by itself.[1] The supergravity gauge fields turn the rigid symmetries of eqs. (2.4) into local ones and transform as follows

$$
\begin{aligned}
\delta e &= \dot{\xi} + i\bar{\chi}^i \epsilon_i + i\chi_i \bar{\epsilon}^i \\
\delta \chi_i &= \dot{\epsilon}_i - i a_i^k \epsilon_k + i\alpha_i^k \chi_k = \mathcal{D}\epsilon_i + i\alpha_i^k \chi_k \\
\delta \bar{\chi}^i &= \dot{\bar{\epsilon}}^i + i a^i_k \bar{\epsilon}^k - i\alpha^i_k \bar{\chi}^k = \mathcal{D}\bar{\epsilon}^i - i\alpha^i{}_k \bar{\chi}^k \\
\delta a^i_j &= \dot{\alpha}^i_j - i a_j^k \alpha^i_k + i a^i_k \alpha^k_j = \mathcal{D}\alpha^i_j
\end{aligned} \tag{2.7}
$$

where $\mathcal{D}$ stands for the U(N) covariant derivative.

From the phase space action (2.6) it is immediate to see that the equations of motion of the gauge fields $G \equiv (e, \chi, \bar{\chi}, a)$ constrain the Noether charges to vanish

$$
\frac{\delta S}{\delta G} = 0 \qquad \Rightarrow \qquad H = Q_i = \bar{Q}^i = J_i^{\,j} - s\delta_i^j = 0\,. \tag{2.8}
$$

The Poisson brackets of these generators form the U(N)-extended supersymmetry algebra that, as we shall see, ceases to be first class for $N > 2$ on arbitrary curved manifolds. This hints to a fundamental obstruction in imposing the constraints listed above and is the signal, from a worldline point of view, of the difficulties that arise in coupling higher spin particles to curved spaces. We will discuss this issue in more depth in section 4.

[1] The Chern-Simons coupling s is taken to be proportional to $\hbar$ and vanish at the classical level, so that the constraint $\psi\bar{\psi} = s\delta$ is also consistent classically when considering ψ and $\bar{\psi}$ as Grassmann variables. At the quantum level the Chern-Simons term contributes together with the quantum effects of the fermions ψ and $\bar{\psi}$, and must be quantized to avoid gauge anomalies. For simplicity in the paper we set $\hbar = 1$.

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Eliminating the momenta p and $\bar{p}$ one obtains the action in configuration space

$$S[X,G]=\int_0^1 d\tau\big[e^{-1}\big(\dot{x}^{\mu}-i\bar{\chi}^{i}\psi_i^{\mu}\big)\big(\dot{\bar{x}}_{\mu}-i\chi_j\bar{\psi}^{j}_{\mu}\big)+i\bar{\psi}^{i}_{\mu}\big(\delta_i^j\partial_\tau-ia_i^j\big)\psi_j^{\mu}+sa_i^i\big] \tag{2.9}$$

where $X\equiv(x,\bar{x},\psi,\bar{\psi})$ and $G\equiv(e,\chi,\bar{\chi},a)$. We shall use this form when constructing the path integral in section 5.

3 Equations of motion in flat space

We now use canonical quantization and obtain the equations of motion in flat space. From the constraint $H=0$, we see that the system has a constant τ evolution. The dynamics of the particle is then entirely contained in the constraints $H=Q_i=\bar{Q}^i=J_i^j-s\delta_i^j=0$: these classical statements translate, in the quantum theory, into the selection of the physical Hilbert space, which is obtained by requiring the symmetry generators to annihilate physical states, i.e.

$$|\Phi\rangle\in\mathcal{H}_{\text{phys}}\quad\Longleftrightarrow\quad T_a|\Phi\rangle=0\,,\quad T_a=(H\,,\,Q_i\,,\,\bar{Q}^i\,,\,J_i^j-s\delta_i^j) \tag{3.1}$$

where the generators T_a are now to be understood as operators. The Chern-Simons coupling s will satisfy a quantization condition that can be stated precisely once a prescription for resolving the ordering ambiguities contained in J_i^j is taken care of. What we have just described is the Dirac quantization procedure, which generalizes the quantization à la Gupta-Bleuler of electrodynamics. As already discussed in [7], the particle states can be represented by generalized field strengths of the form $F_{\mu_1^1\dots\mu_m^1,\dots,\mu_1^N\dots\mu_m^N}$, where the integer m is related to the quantized Chern-Simons coupling s. In particular, the J constraints require that F is antisymmetric within each block of m indices, symmetric in exchanging entire blocks, and in addition satisfies algebraic Bianchi identities, i.e. it belongs to an irreducible representation of U(d) with rectangular $m\times N$ Young tableau:

$$F_{\mu_1^1\dots\mu_m^1,\dots,\mu_1^N\dots\mu_m^N}\quad\sim\quad m\left\{\underbrace{\begin{array}{|c|c|c|c|}\hline &&&\\\hline &&&\\\hline &&&\\\hline\end{array}}_{N}\right. . \tag{3.2}$$

The Q and $\bar{Q}$ constraints enforce generalized Maxwell equations, while the H constraint is automatically satisfied in virtue of the constraint algebra.

We now proceed in deriving the results stated above: looking at the fundamental (anti)-commutation relations, which follows from the classical Poisson brackets (2.3),

$$\big[x^{\mu}\,,\,p_{\nu}\big]=i\delta^{\mu}_{\nu}\,,\quad\big[\bar{x}^{\bar{\mu}}\,,\,\bar{p}_{\bar{\nu}}\big]=i\delta^{\bar{\mu}}_{\bar{\nu}}\,,\quad\big\{\psi_i^{\mu}\,,\,\bar{\psi}^{\bar{\nu}\,j}\big\}=\delta^{\mu\bar{\nu}}\delta_i^j\,, \tag{3.3}$$

one can decide to project the states of the Hilbert space onto the x^{μ}, $\bar{x}^{\bar{\mu}}$ and ψ_i^{μ} eigenstates. In this way x, $\bar{x}$ and ψ act by multiplication, while momenta p, $\bar{p}$ and $\bar{\psi}$ act as derivatives: $p_{\mu}\sim-i\partial_{\mu}$, $\bar{p}_{\bar{\mu}}\sim-i\bar{\partial}_{\bar{\mu}}$ and $\bar{\psi}^{\bar{\mu}i}\sim\frac{\partial}{\partial\psi_i^{\mu}}$. The states are thus represented by functions of x,

$\bar{x}$ and ψ: $|F\rangle \sim \langle x, \bar{x}, \psi | F\rangle = F(x, \bar{x}, \psi)$. With this realization the symmetry generators T_a read

$$\begin{aligned} J_i^{\,j} - s\delta_i^j &= \psi_i \cdot \frac{\partial}{\partial \psi_j} - m\,\delta_i^j \\ Q_i &= -i\psi_i^\mu\,\partial_\mu \\ \bar{Q}^i &= -i\,\frac{\partial}{\partial \psi_i^\mu}\,\bar{\partial}^\mu \\ H &= -\delta^{\mu\bar{\nu}}\,\partial_\mu \bar{\partial}_{\bar{\nu}} \end{aligned} \tag{3.4}$$

where $\bar{\partial}^\nu = \delta^{\nu\bar{\mu}}\,\bar{\partial}_{\bar{\mu}}$. Ordering ambiguities are only present in the J constraint. We have resolved them by using a graded-symmetric ordering, which coincides with the natural regularization that arises form the path integral of section 5,

$$J_i^{\,j} = \frac{1}{2}(\psi_i^\mu \bar{\psi}^j_\mu - \bar{\psi}^j_\mu \psi_i^\mu) = \psi_i^\mu \bar{\psi}^j_\mu - \frac{d}{2}\delta_i^j \quad \Longrightarrow \quad J_i^{\,j} - s\delta_i^j = \psi_i \cdot \frac{\partial}{\partial \psi_j} - m\,\delta_i^j \tag{3.5}$$

where we have set $m \equiv (\frac{d}{2} + s)$. The quantum constraints satisfy an algebra of first class corresponding to the quantum version of (2.5)

$$\begin{aligned} &\{Q_i, \bar{Q}^j\} = \delta_i^j H \\ &[J_i^{\,j}, Q_k] = \delta_k^j Q_i\,, \qquad\qquad [J_i^{\,j}, \bar{Q}^k] = -\delta_i^k \bar{Q}^j \\ &[J_i^{\,j}, J_k^{\,l}] = \delta_k^j J_i^{\,l} - \delta_i^l J_k^{\,j} \end{aligned} \tag{3.6}$$

while other independent graded-commutators vanish. Here we have used the simple $J_i^{\,j}$ generators, but it is evident that the same result holds by substituting them with $J_i^{\,j} - s\delta_i^j$ since the Chern-Simons term is central and in addition it cancels on right hand sides.

Due to the grassmannian nature of the ψ variables, the states have a finite Taylor expansion in ψ's

$$|F\rangle \sim \sum_{A_i=0}^{d} F_{\mu_1^1 \cdots \mu_{A_1}^1, \ldots, \mu_1^N \cdots \mu_{A_N}^N}(x, \bar{x})\, \psi_1^{\mu_1^1} \ldots \psi_1^{\mu_{A_1}^1} \ldots \psi_N^{\mu_1^N} \ldots \psi_N^{\mu_{A_N}^N} \tag{3.7}$$

and we can now study which of them survive the constraint equations.

First we consider the $J_i^{\,j}$ constraints. The $J_i^{\,i}$ constraint at fixed i counts fermions of i-th type and fixes them to be m in number, see (3.5). Thus m must be an integer and this, in turn, fixes the possible quantized values of the Chern-Simons coupling s. Hence, the only term of (3.7) surviving this constraint is

$$F_{\mu_1^1 \cdots \mu_m^1, \ldots, \mu_1^N \cdots \mu_m^N}\, \psi_1^{\mu_1^1} \ldots \psi_1^{\mu_m^1} \ldots \psi_N^{\mu_1^N} \ldots \psi_N^{\mu_m^N} \tag{3.8}$$

i.e. a tensor with N blocks of m indices. In term of complex geometry, the tensor $F_{\mu_1^1 \cdots \mu_m^1, \ldots, \mu_1^N \cdots \mu_m^N}(x, \bar{x})$ can be thought of a differential multiple $(m, 0)$-form: in fact each ψ_i block in (3.8) plays the role of a basis for the $(m, 0)$-forms, $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_m}$. The $J_i^{\,j}$ constraint for $i \neq j$ then ensures algebraic Bianchi identities: it picks an index of the j-th block, antisymmetrizes it with those of the i-th block, and set the resulting tensor to zero. For example, the J_1^2 constraint gives

$$F_{[\mu_1^1 \dots \mu_m^1, \mu_1^2] \dots, \mu_1^N \dots \mu_m^N} = 0 \tag{3.9}$$

and so on. As a consequence, the tensor $F_{\mu_1^1 \dots \mu_m^1, \dots, \mu_1^N \dots \mu_m^N}$ has N blocks of m antisymmetric indices and is symmetric under exchanges of blocks. The antisymmetry within each block is evident from the Grassmann nature of the ψ's, while symmetry between blocks can be understood considering particular U(N) transformations. In fact, a $\frac{\pi}{2}$ rotation in the $i-j$ plane sends ψ_i in ψ_j and ψ_j in $-\psi_i$. The final effect of this U(N) transformation is to exchange the i-th and j-th blocks of indices on the tensor F in (3.8) without any additional sign. Since this is a U(N) transformation connected to the identity, it can be cast in the form $e^{i\alpha_j^i J_i^j}$ for some α_j^i with $i \neq j$. Requiring $J_i^j|F\rangle = 0$ produces the anticipated symmetry between the i-th and j-th blocks of indices of the tensor F. All these algebraic symmetries are summarized by saying that F belongs to an irreducible representation of the group U(d) described by the Young tableau in eq. (3.2). Finally, using the representation (3.4) of the operators Q_i and $\bar{Q}^i$, it is straightforward to see that their constraints impose the following generalized Maxwell equations on the curvature F

$$\partial_{[\mu} F_{\mu_1^1 \dots \mu_m^1], \dots, \mu_1^N \dots \mu_m^N} = 0 \, , \tag{3.10}$$

$$\bar{\partial}^{\mu} F_{\mu \dots \mu_m^1, \dots, \mu_1^N \dots \mu_m^N} = 0 \, . \tag{3.11}$$

3.1 Gauge fields

In analogy with Maxwell, Yang-Mills and higher spin gauge theories, we first try to solve eq. (3.10). This equation can be interpreted as an integrability condition. In the absence of topological obstructions, the closure of a form F is achieved expressing it as the exterior derivative of a gauge field: $F = d\phi \ \rightarrow \ dF = 0$. In our context we are dealing with N-multiple $(m, 0)$-forms, and we are going to show that (3.10), that is to say $Q_i|F\rangle = 0$, can be solved writing F as the multiple action (one for each block of indices) of the holomorphic Dolbeault operator ∂, that sends forms of bidegree (p, q) into $(p+1, q)$-forms. As the $\partial_{(i)}$ operator[2] in our quantum mechanical notation is simply Q_i, it is useful to define

$$q = Q_1 Q_2 \dots Q_N \tag{3.12}$$

which is identically annihilated by the Q_i's: $qQ_i = Q_i q = 0$, due to $Q_i^2 = 0$ and to the fact that q contains already all of the Q_i's. Setting

$$|F\rangle = q|\phi\rangle \tag{3.13}$$

automatically satisfies the Q constraints and, writing down (3.13) in components, we see that $F \sim \partial_{(1)} \dots \partial_{(N)}\phi$, where each Dolbeault operator antisymmetrizes only over the

[2]The index i refers to the block on which the Dolbeault operator ∂ acts, while other blocks are treated as spectators.

corresponding block of indices. To solve the J constraints one can take ϕ to be a N-multiple $(p,0)$-form with $p \equiv m-1$ that forms a $\mathrm{U}(d)$ irreducible tensor (a rectangular $p \times N$ Young tableau)

$$|\phi\rangle \quad \sim \quad \phi_{\mu_1^1\ldots\mu_p^1,\ldots,\mu_1^N\ldots\mu_p^N}(x,\bar{x}) \quad \sim \quad p\left\{\underbrace{\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline \end{array}}_{N}\right. . \tag{3.14}$$

In fact, we note that $J_i^j\, q = q\,(J_i^j + \delta_i^j)$. Thus $(J_i^j - s\delta_i^j)|F\rangle = 0$ is satisfied if one requires

$$\big(J_i^j - (s-1)\delta_i^j\big)\,|\phi\rangle = 0\ , \tag{3.15}$$

that is, $N_i = m-1 \equiv p$ if taking $i=j$ (by $N_i \equiv \psi_i \cdot \frac{\partial}{\partial \psi_i}$ at fixed i we indicate the number operator that counts the fermions of the i-th type), while the off diagonal equations are the same as for F: they impose algebraic Bianchi identities and, in particular, symmetry between block exchanges.

Next it remains to implement the last independent constraint, $\bar{Q}^i|F\rangle = 0$. This produces generalized Maxwell equations for the gauge field. From (3.13) it is clear that $\bar{Q}^i\, q|\phi\rangle = 0$ is an higher derivative equation of motion for the gauge potential, precisely of order $N+1$. It is convenient to use some $Q, \bar{Q}$ algebra in order to factorize from the operator $\bar{Q}^i\, q$ a second order differential operator G, that will play a role analogous to the Fronsdal-Labastida operator [8, 13] for higher spin fields. Iterated use of $\{Q_i, Q_j\} = 0$ and $\{Q_i, \bar{Q}^j\} = \delta_i^j\, H$ gives (in the following equation j is fixed, not summed)

$$\begin{aligned}\bar{Q}^j\, q =& \bar{Q}^j\, Q_1 Q_2 \ldots Q_N = (-1)^{j-1} Q_1 \ldots \bar{Q}^j Q_j \ldots Q_N \\ =& (-1)^{j-1}\big(Q_1 \ldots Q_{j-1} Q_{j+1} \ldots Q_N\big)\bar{Q}^j Q_j \\ =& (-1)^{j-1}\big(Q_1 \ldots Q_{j-1} Q_{j+1} \ldots Q_N\big)\big(H - Q_j \bar{Q}^j\big)\ .\end{aligned}$$

At this point is possible to sum over j in $H - Q_j\bar{Q}^j$, since the extra terms vanish anyhow, and cast the equation of motion in the form

$$\bar{Q}^j q|\phi\rangle = q^j G|\phi\rangle = 0 \tag{3.16}$$

where, in an obvious notation, $q^j \equiv (-1)^j Q_1 \ldots Q_{j-1} Q_{j+1} \ldots Q_N$. G is the second order operator we were looking for, analogous to the Fronsdal-Labastida operator without the trace term

$$G = -H + Q_i \bar{Q}^i \quad \sim \quad \partial_\alpha \bar{\partial}^\alpha - \psi_i^\alpha\, \frac{\partial}{\partial \psi_i^\beta}\, \partial_\alpha \bar{\partial}^\beta\ . \tag{3.17}$$

To obtain a second order equation of motion from (3.16) it is necessary to eliminate the operator q^j. One way to do this is recalling that a generic expression containing two Q's represents the kernel of q^j, that is $q^j Q_k Q_l \equiv 0$, and so a general solution of $q^i\big(G|\phi\rangle\big) = 0$ is

$$G|\phi\rangle = Q_i Q_j |\rho^{ij}\rangle \tag{3.18}$$

where $|\rho^{ij}\rangle$ are the compensator fields. One can present the compensators also in the form $|\rho^{ij}\rangle = \bar{V}^i \bar{V}^j |\rho\rangle$. This second form of writing the compensators is slightly more convenient.

Here $\bar{V}^i \equiv V^\mu \bar{\psi}^i_\mu$ depends on an arbitrary vector field V^μ, and $|\rho\rangle$ is a state that must satisfy $(J^j_i - (s-1)\delta^j_i)|\rho\rangle = 0$ (because of eq. (3.15) and $[G, J^j_i] = 0$) and thus is represented by a tensor with the same structure and Young tableau of ϕ. The action of $\bar{V}^i$ is to eliminate one ψ from the i-th block and saturate the corresponding index of the ρ tensor with V^μ. Therefore the compensator ρ^{ij} has $N-2$ blocks with p antisymmetric indices and two blocks, the i-th and j-th ones, with $p-1$ indices. Its Young tableau has the form

$$\rho^{ij} \quad \sim \quad p\left\{ \underbrace{\begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & \\ \cline{1-2} \end{array}}_{N} \right. . \tag{3.19}$$

The key feature of eq. (3.18) is to be a second order wave equation. The price for this is the introduction of the auxiliary fields ρ^{ij}. Of course one would like also to obtain an equation without compensators, $G|\phi\rangle = 0$. This is indeed possible using gauge symmetries. In fact, in theories where the physical field strength is expressed in terms of a potential, one expects the presence of a gauge symmetry.

Gauge symmetry. In term of forms if $F = d\phi$, the gauge transformation leaving F invariant is $\delta\phi = d\Lambda$. In our model the gauge symmetry enjoyed by the curvature F is an "higher spin" generalization of the linearized diffeomorphisms of general relativity, like the gauge transformations of standard higher spin fields. In our operator formalism, exterior holomorphic derivatives acting on the i-th block are represented by the supercharge Q_i. Thus, recalling that $|F\rangle = q|\phi\rangle$ and $qQ_i = 0$, one finds immediately an invariant way of writing down the gauge transformations that leave the F tensor invariant

$$\delta|\phi\rangle = Q_i|\Lambda^i\rangle \tag{3.20}$$

where $|\Lambda^i\rangle$ are the gauge parameters. Again a slightly more convenient way of writing the gauge parameters is in the form $|\Lambda^i\rangle = \bar{W}^i|\Lambda\rangle$, where $\bar{W}^i \equiv W^\mu \bar{\psi}^i_\mu$ with W^μ a vector field and $|\Lambda\rangle$ a state containing a tensor with the same index structure and Young tableau of $|\phi\rangle$. These gauge transformations clearly do not affect $|F\rangle = q|\phi\rangle$, but let us compute how the left hand side of (3.18) transforms. Making use of the Q, $\bar{Q}$ algebra the gauge variation can be written as

$$G\delta|\phi\rangle = -Q_i Q_j \left(\bar{Q}^i|\Lambda^j\rangle\right) , \tag{3.21}$$

and if we want the equations of motion to be gauge invariant, the compensator field (from this its name) has to cancel the above expression and transform as

$$\delta|\rho^{ij}\rangle = -\bar{Q}^{[i}|\Lambda^{j]}\rangle . \tag{3.22}$$

It is well known from higher spin field theories [9, 12] that the equations of motion in the compensator formalism are invariant for general gauge transformations, but if we try to gauge fix the compensators to zero, constraints on gauge parameters and on gauge fields appear, namely the gauge parameters must be traceless and the gauge fields double traceless. In our framework there are no ways of taking the trace of completely holomorphic

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tensors, instead differential constraints appear on gauge parameters and on gauge fields. To see this, let us use part of the gauge freedom in (3.20) and (3.22) to make the compensators vanish: $\rho^{ij} = 0$. The residual gauge symmetry must satisfy $\bar{Q}^{[i}|\Lambda^{j]}\rangle = 0$, and this can be achieved if the gauge parameters are taken to be "divergenceless": $\bar{Q}^i|\Lambda^j\rangle = 0$ for $i \neq j$. Similarly, the gauge choice $\rho^{ij} = 0$ imposes constraints also on the gauge field ϕ. This can be seen by acting with $\bar{Q}^k$ on both sides of eq. (3.18) to obtain

$$Q_i \bar{Q}^k \bar{Q}^i |\phi\rangle = Q_i \left[\bar{Q}^k Q_j |\rho^{ij}\rangle - H|\rho^{ki}\rangle \right] . \tag{3.23}$$

The right hand side of this equation vanishes in the partially gauge fixed theory with $\rho^{ij} = 0$. For consistency the left hand side must vanish as well, and this is guaranteed if $\bar{Q}^k \bar{Q}^i|\phi\rangle = 0$, that corresponds to setting to zero all possible double divergences. One may check that this constraint is kept invariant by gauge transformations with parameters satisfying $\bar{Q}^i|\Lambda^j\rangle = 0$ with $i \neq j$. Once the compensator fields have been eliminated, the gauge potential describing the particle satisfies the simpler second order wave equation $G|\phi\rangle = 0$ that, in tensorial language, reads

$$\partial_\alpha \bar{\partial}^\alpha \phi_{\mu_1\ldots\mu_p,\ldots,\nu_1\ldots\nu_p} - p\, \partial_{\mu_1} \bar{\partial}^\alpha \phi_{\alpha\mu_2\ldots\mu_p,\ldots,\nu_1\ldots\nu_p} - \ldots - p\, \partial_{\nu_1} \bar{\partial}^\alpha \phi_{\mu_1\ldots\mu_p,\ldots,\alpha\nu_2\ldots\nu_p} = 0 \tag{3.24}$$

where $p \equiv m - 1$, and weighted antisymmetrization is understood on μ's, ν's and so on.

In order to clarify the meaning of our quantum mechanical notation, let us analyze in tensorial language a specific case: $N = 2$, $p = 2$. This is the simplest model where all of the issues treated so far appear in a non trivial way. The gauge field ϕ has the structure

$$\phi_{\mu_1\mu_2,\nu_1\nu_2} \sim \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \tag{3.25}$$

while the unique independent compensator is a symmetric tensor $\rho_{\mu\nu}$. The gauge invariant equations of motion read

$$\partial_\alpha \bar{\partial}^\alpha \phi_{\mu_1\mu_2,\nu_1\nu_2} - 2\partial_{\mu_1} \bar{\partial}^\alpha \phi_{\alpha\mu_2,\nu_1\nu_2} - 2\partial_{\nu_1} \bar{\partial}^\alpha \phi_{\mu_1\mu_2,\alpha\nu_2} = 2\partial_{\mu_1}\partial_{\nu_1}\rho_{\mu_2\nu_2} \tag{3.26}$$

with an understood weighted antisymmetrization on the μ and ν group of indices, that will be employed in all of the following equations as well. The gauge transformations for the field ϕ and the compensator are given by

$$\delta\phi_{\mu_1\mu_2,\nu_1\nu_2} = \partial_{\mu_1}\Lambda_{\nu_1\nu_2,\mu_2} + \partial_{\nu_1}\Lambda_{\mu_1\mu_2,\nu_2} \ , \quad \delta\rho_{\mu\nu} = -\bar{\partial}^\alpha \Lambda_{\alpha\mu,\nu} - \bar{\partial}^\alpha \Lambda_{\alpha\nu,\mu} \tag{3.27}$$

where a factor of $-2i$ has been absorbed in the definition of the gauge parameter, whose Young tableau is

$$\Lambda_{\mu_1\mu_2,\nu} \sim \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \ . \tag{3.28}$$

Using part of the gauge freedom, one can fix the compensator to zero, obtaining the gauge invariant equation

$$\partial_\alpha \bar{\partial}^\alpha \phi_{\mu_1\mu_2,\nu_1\nu_2} - 2\partial_{\mu_1} \bar{\partial}^\alpha \phi_{\alpha\mu_2,\nu_1\nu_2} - 2\partial_{\nu_1} \bar{\partial}^\alpha \phi_{\mu_1\mu_2,\alpha\nu_2} = 0 \tag{3.29}$$

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which is left invariant by the gauge transformations (3.27) with constrained gauge parameters

$$\bar{\partial}^{\alpha}\Lambda_{\alpha\mu,\nu} = 0 \; . \tag{3.30}$$

For consistency, the gauge field appearing in this equation must also satisfy a differential constraint

$$\bar{\partial}^{\alpha}\bar{\partial}^{\beta}\phi_{\alpha\mu,\beta\nu} = 0 \tag{3.31}$$

which is preserved by the gauge transformations with constrained gauge parameters.

To count the physical degrees of freedom, one has to use the remaining gauge freedom to eliminate unphysical "polarizations" from ϕ. This way one ends up with a gauge field $\phi_{m_1\ldots m_p,\ldots,n_1\ldots n_p}$, where indices run over $d-2$ directions, i.e. $m, n = 1, 2, \ldots, d-2$. Perhaps this is best seen in the particle language, since by using the complex N supersymmetries one can eliminate the fermionic fields ψ_i^{μ} and their complex conjugates with the index μ pointing along two chosen directions. In this "light cone gauge" the tensor $\phi_{m_1\ldots m_p,\ldots,n_1\ldots n_p}$ describes an irreducible representation of the little group for massless particles, $\mathrm{U}(d-2)$, with the same Young tableau of eq. (3.14). The dimension of such representation corresponds to the number of physical degrees of freedom of the particle. Using the "factors over hook" rule it is easy to compute the dimension of this Young tableau, and the resulting degrees of freedom, for all d, N and p, are

$$Dof(d, N, p) = \prod_{j=0}^{N-1} \frac{j!(j+d-2)!}{(j+p)!(j+d-2-p)!} \tag{3.32}$$

where we recall that $p = m - 1 = \frac{d}{2} + s - 1$. We note that in the case of an odd number of complex dimensions the physical spectrum is empty unless the Chern-Simons term is added, i.e. $s \neq 0$. The quantization of this Chern-Simons coupling can be understood also from the requirement of cancelling gauge anomalies [14].

Let us analyze a few examples. From (3.32) one can see that in $d = 2$ (four *real* dimensions) without Chern-Simons coupling, there is always one degree of freedom for any value of N: $Dof(2, N, 0) = 1$. So, with $s = 0$, all the $\mathrm{U}(N)$ spinning particle theories propagate only a scalar field in two complex dimensions, and share for this aspect the features of $N = 2$ superstrings, where only the scalar ground states survive at the critical dimension $d = 2$, see for example the review [15]. Another simple case is the $N = 1$ theory in arbitrary complex dimensions: the field strengths are $(p+1, 0)$-forms $F_{\mu_1\ldots\mu_{p+1}}$, the gauge potentials are $(p, 0)$-forms $\phi_{\mu_1\ldots\mu_p}$ and (3.32) gives $Dof(d, 1, p) = \binom{d-2}{p}$; that is the number of independent components of an antisymmetric tensor of $\mathrm{U}(d-2)$ with p indices, $\phi_{m_1\ldots m_p}$. In the last section we will compute the one-loop partition function for the $\mathrm{U}(N)$ spinning particle. After covariantly gauge fixing the action (2.9) on the torus, the path integral reduces to an integral over a corresponding moduli space which computes the number of physical degrees of freedom. Indeed, we shall see that they coincide with the canonical computation just presented.

To summarize, we have described gauge invariant equations with compensators

$$G|\phi\rangle = Q_i Q_j|\rho^{ij}\rangle \tag{3.33}$$

with $G = -H + Q_i \bar{Q}^i$, and gauge symmetries given by

$$\delta|\phi\rangle = Q_i|\Lambda^i\rangle\,, \qquad \delta|\rho^{ij}\rangle = -\bar{Q}^{[i}|\Lambda^{j]}\rangle \tag{3.34}$$

where $|\rho^{ij}\rangle \equiv \bar{V}^i\bar{V}^j|\rho\rangle$, $|\Lambda^i\rangle \equiv \bar{W}^i|\Lambda\rangle$ and with $|\phi\rangle$, $|\rho\rangle$, $|\Lambda\rangle$ describing tensors with rectangular $p \times N$ Young tableaux of $\mathrm{U}(d)$, as in (3.14).

Similarly, gauge invariant equations without compensators are given by

$$G|\phi\rangle = 0 \tag{3.35}$$

with gauge symmetry

$$\delta|\phi\rangle = Q_i|\Lambda^i\rangle \tag{3.36}$$

where $|\Lambda^i\rangle \equiv \bar{W}^i|\Lambda\rangle$, with fields and gauge parameters satisfying the differential constraints

$$\bar{Q}^i\bar{Q}^j|\phi\rangle = 0\,, \qquad \bar{Q}^i|\Lambda^j\rangle = 0 \qquad (i \neq j). \tag{3.37}$$

4 Supersymmetry algebra in curved Kähler manifolds

We now turn to study the supersymmetry algebra on arbitrary Kähler manifolds. It will be shown that for all N it is possible to close the algebra, though quadratically, on Kähler manifolds of constant holomorphic curvature, and so even for $N > 2$ a consistent quantization can be obtained beyond the case of flat space.

Looking at the quantum algebra (3.6), we note that the last three relations just state that $J_i^{\,j}$ are $\mathrm{U}(N)$ generators and that $Q_i, \bar{Q}^j$ belong to the $\mathbf{N}$, $\bar{\mathbf{N}}$ of $\mathrm{U}(N)$, and presumably these relations should be left unchanged even in curved space. The first equation is the key ingredient of the supersymmetry algebra, and is going to be modified by a nonvanishing curvature. Our aim is to deform the algebra (3.6) introducing curvature, but keeping it first class, as necessary if we want to impose the corresponding constraints consistently.

Thus, let us consider the theory on an arbitrary Kähler manifold. The only non vanishing components of the metric are $g_{\mu\bar{\nu}}(x,\bar{x}) = g_{\bar{\nu}\mu}(x,\bar{x})$, which lead to nonvanishing Christoffel coefficients for the total holomorphic or antiholomorphic parts only: $\Gamma^{\mu}_{\nu\lambda}$, $\Gamma^{\bar{\mu}}_{\bar{\nu}\bar{\lambda}}$. In curved space we will use fermions with flat indices: ψ_i^a and $\bar{\psi}^{\bar{a}i}$; the $\mathrm{U}(N)$ generators are essentially unchanged, being defined by $J_i^{\,j} = \frac{1}{2}\big[\psi_i^a\,, \bar{\psi}_a^j\big]$ (the flat tangent metric is simply $\delta_{a\bar{b}}$), but the supercharges need a suitable covariantization. Since the holonomy group of Kähler manifolds of real dimension $D = 2d$ is $\mathrm{U}(d)$, the connection would be a $\mathrm{U}(d)$ spin connection, and the covariant derivative reads

$$\nabla_\mu = \partial_\mu + \omega_{\mu a\bar{b}} M^{a\bar{b}}$$

where $M^{a\bar{b}}$ are the $\mathrm{U}(d)$ generators. In the particle model these generators can be realized by

$$M^{a\bar{b}} = \frac{1}{2}\big[\psi_i^a\,, \bar{\psi}^{\bar{b}\,i}\big] = \psi_i^a\bar{\psi}^{\bar{b}i} - \frac{N}{2}\delta^{a\bar{b}} \tag{4.1}$$

as they satisfy indeed the Lie algebra of $\mathrm{U}(d)$

$$\big[M^{a\bar{b}}\,, M^{c\bar{d}}\big] = \delta^{c\bar{b}}M^{a\bar{d}} - \delta^{a\bar{d}}M^{c\bar{b}}\,.$$

In this way we construct covariantized momenta[3]

$$\begin{aligned} \pi_\mu &= g^{1/2}\big(p_\mu - i\omega_{\mu a\bar b} M^{a\bar b}\big) g^{-1/2} \\ \bar\pi_{\bar\mu} &= g^{1/2}\big(\bar p_{\bar\mu} - i\omega_{\bar\mu a\bar b} M^{a\bar b}\big) g^{-1/2} , \end{aligned} \tag{4.2}$$

and supercharges

$$Q_i = \psi_i^a \, e_a^\mu \, \pi_\mu \quad , \quad \bar Q^j = \bar\psi^{\bar a j} \, e_{\bar a}^{\bar\mu} \, \bar\pi_{\bar\mu} . \tag{4.3}$$

With these charges the JJ, JQ and $J\bar Q$ commutators are the same as before, but the $Q\bar Q$ anticommutator now reads

$$\{Q_i , \bar Q^j\} = \delta_i^j \, H_0 - R_{a\bar b c\bar d} \, \psi_i^a \bar\psi^{\bar b j} M^{c\bar d} , \tag{4.4}$$

where $H_0 = g^{\bar\mu\nu}\bar\pi_{\bar\mu}\pi_\nu$ is the minimal covariantization of the hamiltonian. As in the case of $O(N)$ supersymmetry [5, 16], we can achieve the closure of the algebra on particular manifolds, namely Kähler manifolds with constant holomorphic curvature, which admit a Riemann tensor of the form [17]

$$R_{a\bar b c\bar d} = \Lambda \left(\delta_{a\bar b}\delta_{c\bar d} + \delta_{a\bar d}\delta_{c\bar b}\right) , \tag{4.5}$$

with constant Λ. As for real manifolds maximally symmetric spacetimes are de Sitter, anti-de Sitter and flat Minkowski space, prototypes of Kähler manifolds with a Riemann tensor of the form (4.5) are the complex projective space $\mathbb{CP}^d$, complex hyperbolic space $\mathbb{CH}^d$ and, of course, flat complex space $\mathbb{C}^d$ viewed as a Kähler manifold. Inserting the $\mathrm{U}(d)$ generators $M^{a\bar b} = \frac{1}{2}\big[\psi_i^a , \bar\psi^{\bar b\, i}\big]$, the $\{Q, \bar Q\}$ anticommutator closes quadratically (up to an obvious redefinition of the hamiltonian)

$$\{Q_i , \bar Q^j\} = \delta_i^j \left(H_0 - a\, J - b\right) - \Lambda J_i^j \, J + \frac{\Lambda}{2}\,\{J_i^k , J_k^j\} , \tag{4.6}$$

with $J = J_k^k$, $a = \Lambda\frac{d+1}{2}$ and $b = \Lambda\frac{d(N+d)}{4}$. The hamiltonian H_0 has, however, an unusual commutator with the supercharges, namely

$$\begin{aligned} \big[H_0 , Q_i\big] &= -\Lambda \, J_i^k \, Q_k + \Lambda \, J \, Q_i + \Lambda \frac{N+d}{2} \, Q_i \quad , \\ \big[H_0 , \bar Q^i\big] &= -\big[H_0 , Q_i\big]^\dagger , \end{aligned} \tag{4.7}$$

so we add to H_0 a hermitian and $\mathrm{U}(N)$ neutral J combination in order to cancel the commutators above. We recall that, including a Chern-Simons coupling, the quantum constraint on J is $J_i^j - s\delta_i^j = 0$ and so, in order to make manifest the quadratic closure of our algebra, we set $\tilde J_i^j = J_i^j - s\delta_i^j$ and $\tilde J = \tilde J_i^i$, finally obtaining

$$\begin{aligned} &\big[H , \tilde J_i^j\big] = \big[H , Q_i\big] = \big[H , \bar Q^j\big] = 0 \\ &\big[\tilde J_i^j , \tilde J_k^l\big] = \delta_k^j \, \tilde J_i^l - \delta_i^l \, \tilde J_k^j \\ &\big[\tilde J_i^j , Q_k\big] = \delta_j^k \, Q_i \quad , \qquad\qquad \big[\tilde J_i^j , \bar Q^k\big] = -\delta_i^k \, \bar Q^j \\ &\{Q_i , \bar Q^j\} = \delta_i^j H + \Lambda\left[\tilde J_i^k \, \tilde J_k^j - \tilde J_i^j \tilde J + h_1 \tilde J_i^j + \frac{1}{2}\delta_i^j\left(\tilde J^2 - \tilde J_k^l \tilde J_l^k + h_2 \tilde J\right)\right] , \end{aligned} \tag{4.8}$$

[3] We denote $g = \det g_{\mu\bar\nu}$ and the g factors ensure hermiticity.

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where the complete hamiltonian reads

$$H = H_0 + \frac{\Lambda}{2}\left[J_i^k J_k^i - J^2 - h_3 J - h_4\right] , \tag{4.9}$$

with the h_i being defined by

$$\begin{aligned} h_1 &= (2-N)s - \frac{N}{2} , \\ h_2 &= 2s(N-2) + 1 , \qquad h_3 = d + 1 , \\ h_4 &= \frac{d}{2}(N+d) - s^2(N-1)(N-2) . \end{aligned} \tag{4.10}$$

This is no more a Lie algebra but, being still first class, permits a consistent realization of the constraints $\tilde{J}_i^j = H = Q_i = \bar{Q}^j = 0$, which define higher spin equations on such curved backgrounds. As the analogous result obtained in [5] for the $O(N)$ spinning particle, the quadratic algebra (4.8) coincides with the zero mode, in the Ramond sector, of the quadratic U(N) superconformal algebra found by Bershadsky and Knizhnik in [10, 11].

Up to now we have used U(d) generators with the preferred ordering given in (4.1), but a quadratic closure of the supersymmetry algebra can be achieved with an arbitrary ordering, corresponding to a different coupling to the U(1) part of the spin connection $\omega_\mu = \omega_{\mu a\bar{b}}\delta^{a\bar{b}}$: if in eq. (4.2) we choose as U(d) generators

$$\mathcal{M}^{a\bar{b}} = \psi_i^a \bar{\psi}^{\bar{b}i} - c\delta^{a\bar{b}} , \tag{4.11}$$

with arbitrary c, (4.4) remains unchanged in form, and choosing the Riemann tensor as in (4.5), the quadratic algebra in (4.8) and (4.9) maintains the same structure but with different numerical coefficients $h_i \to h_i(c)$, given by

$$\begin{aligned} h_1(c) &= (2-N)s - \frac{d}{2}(N-2c) + c - N \\ h_2(c) &= (d+1)(N-2c) + 2s(N-2) + 1 \\ h_3(c) &= (d+1)(N-2c+1) \\ h_4(c) &= d\left[\frac{d}{2}(N-2c+1) + N - c\right] + s(N-1)\left[(d+1)(2c-N) - s(N-2)\right] . \end{aligned} \tag{4.12}$$

To recover the previous results is sufficient to put $c = N/2$ in the above formulas.

With this constraint algebra at hand it is possible to achieve the quantization of the U(N) particle, for all N, on Kähler manifolds of constant holomorphic curvature. We expect that a consistent quantization can be achieved also on more general Kähler manifold, namely those possessing a vanishing Bochner tensor, a Kähler analogue of the conformal Weyl tensor, but we have not worked out the explicit constraint algebra.

5 Partition function and degrees of freedom

In order to extract from the U(N) spinning particle action (2.9) the number of physical excitations, we proceed in computing the one-loop partition function that gives, as its first

Seeley-DeWitt coefficient, the number of degrees of freedom. Of course, other heat kernel coefficients vanish in flat space, but once the measure over the moduli space arising from the gauge fixing procedure is correctly identified, one could perform, in principle, more general path integral calculations to investigate the quantum properties of the field equations on the backgrounds described previously.

In order to deal with gaussian path integrals rather than oscillating ones, we perform as usual a Wick rotation on the proper time $\tau \to -i\tau$ and on the gauge field $a^i_j \to ia^i_j$. The resulting euclidean action reads

$$S[X,G] = \int_0^1 d\tau \, \Big[e^{-1}\big(\dot{x}^\mu - \bar{\chi}^i \psi^\mu_i\big)\big(\dot{\bar{x}}_\mu - \chi_j \bar{\psi}^j_\mu\big) + \bar{\psi}^i_\mu \big(\delta^j_i \partial_\tau - i a^j_i\big)\psi^\mu_i - i s a^i_i \Big] \tag{5.1}$$

and is invariant under the supergravity transformations in euclidean time

$$\begin{aligned} \delta e &= \dot{\xi} + \bar{\chi}^i \epsilon_i + \chi_i \bar{\epsilon}^i \\ \delta \chi_i &= \dot{\epsilon}_i - i a^k_i \epsilon_k + i \alpha^k_i \chi_k \\ \delta \bar{\chi}^i &= \dot{\bar{\epsilon}}^i + i a^i_k \bar{\epsilon}^k - i \alpha^i_k \bar{\chi}^k \\ \delta a^i_j &= \dot{\alpha}^i_j - i a^k_j \alpha^i_k + i a^i_k \alpha^k_j \ . \end{aligned} \tag{5.2}$$

The partition function is obtained by performing the functional integral on a circle, taking periodic boundary conditions for the bosonic fields, and antiperiodic ones for the fermionic fields

$$Z = \int_{S^1} \frac{DXDG}{\text{Vol(Gauge)}} \, e^{-S[X,G]} \tag{5.3}$$

where, in condensed notation, $X \equiv (x, \bar{x}, \psi, \bar{\psi})$ refers to the matter fields, while $G \equiv (e, \chi, \bar{\chi}, a)$ represents the supergravity multiplet. Since our model is a gauge theory, it is necessary to divide by the volume of the gauge group. The gauge fixing procedure can be achieved with the standard Faddeev-Popov method. We select a covariant gauge by imposing gauge fixing conditions on the worldline supergravity fields. The latter can be gauged away, except for a remaining finite number of modular integrations that take into account gauge inequivalent configurations. We follow the same strategy employed in [4] for the $O(N)$ spinning particle, to which we refer for additional details.

Gauge fixing on the circle. The einbein $e(\tau)$ has periodic boundary conditions and is characterized by the gauge invariant quantity $\beta = \int_0^1 e(\tau) d\tau$, which represents the invariant length of the circle. A standard gauge for worldline reparametrizations is to fix $e(\tau) = \beta$, and the path integral over e reduces to an ordinary integral over the usual proper time β, with the familiar "one-loop" measure

$$\int_0^\infty \frac{d\beta}{\beta} \, .$$

Due to antiperiodic boundary conditions, the complex gravitini χ_i and $\bar{\chi}^i$ can be completely gauged away, $\chi_i(\tau) = \bar{\chi}^i(\tau) = 0$, leaving corresponding Faddeev-Popov determinants of the differential operators that can be extracted from (5.2). Finally, the gauge field a^j_i can have nontrivial Wilson loops around the circle, that capture the complete gauge invariant information contained in them. They can be gauge fixed to a constant hermitian $N \times N$ matrix, $a_i^j(\tau) = \theta_i^j$, that can be always diagonalized through a constant U(N) gauge transformation

$$\theta_i^j \to \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_N \end{pmatrix} . \tag{5.4}$$

Recalling that a_i^j belongs to the Lie algebra of U(N), we see by exponentiation that the θ_i are in fact angles ranging from 0 to 2π. Now, the path integral over x and $\bar{x}$ gives as usual $V\,(2\pi\beta)^{-d}$, where $V = i^d \int d^d x_0 d^d \bar{x}_0$ (the integral over the x zero modes) is the spacetime volume. The $D\psi D\bar{\psi}$ integral gives $\mathrm{Det}_A(\delta_i^j \partial_\tau - i\theta_i^j)^d$, while integrals over the susy ghosts give a power -2 of the same determinant. Subscripts P and A keep track of the periodic or antiperiodic boundary conditions. From the diagonalization (5.4), we see that the integration over the moduli space of a_i^j reduces to integration over the angles

$$\frac{1}{N!} \prod_{i=1}^{N} \int_0^{2\pi} \frac{d\theta_i}{2\pi} , \tag{5.5}$$

and division by $N!$ is needed to eliminate the overcounting due to the permutations of the θ's, that are all gauge equivalent. The last integration to be performed is over the ghosts for the gauge group U(N), that gives $\mathrm{Det}'_P(\partial_\tau + i\theta_{\mathrm{adj}})$, i.e. with the zero modes removed and the gauge fixed a_i^j taken in the adjoint representation, as follows from $\delta a_i^k = \mathcal{D}\alpha_i^k$ in (5.2). Now, we use the diagonalized form (5.4), and putting together the various contributions we obtain for the partition function

$$\begin{aligned} Z \propto V \int_0^\infty \frac{d\beta}{\beta} \frac{1}{(2\pi\beta)^d} \frac{1}{N!} \prod_{i=1}^{N} \int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{-is\theta_i} \mathrm{Det}_A(\partial_\tau - i\theta_i)^{d-2} \\ \times \prod_{k \neq l} \mathrm{Det}_P(\partial_\tau - i(\theta_k - \theta_l)) . \end{aligned} \tag{5.6}$$

These determinants are standard ones and can be computed using operator methods with simple fermionic systems. Namely, they are: $\mathrm{Det}_A(\partial_\tau - i\theta) = 2\cos\frac{\theta}{2}$ and $\mathrm{Det}_P(\partial_\tau - i\theta) = 2i\sin\frac{\theta}{2}$. Substituting in the expression for Z one finally finds

$$Z \propto V \int_0^\infty \frac{d\beta}{\beta} \frac{1}{(2\pi\beta)^d} \left[\frac{1}{N!} \prod_{i=1}^{N} \int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{-is\theta_i} \left(2\cos\frac{\theta_i}{2}\right)^{d-2} \prod_{k<l} \left(2\sin\frac{\theta_k - \theta_l}{2}\right)^2 \right] . \tag{5.7}$$

Degrees of freedom. The part in square brackets of the above formula gives the number of degrees of freedom of the particle, since the rest is simply the partition function for the center of mass, and so we have the following expression for the physical degrees of freedom

$$Dof(d, N; s) = \frac{1}{N!} \prod_{i=1}^{N} \int_0^{2\pi} \frac{d\theta_i}{2\pi}\, e^{-is\theta_i} \left(2\cos\frac{\theta_i}{2}\right)^{d-2} \prod_{k<l} \left(2\sin\frac{\theta_k - \theta_l}{2}\right)^2 . \tag{5.8}$$

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It is normalized to $Dof(d,0;0) = 1$ for $N = 0$, which corresponds to a simple scalar field. It is now convenient to go to complex coordinates: $z_i = e^{i\theta_i}$. Recalling that $s = m - \frac{d}{2} = p + 1 - \frac{d}{2}$, the above expression in terms of p becomes

$$Dof(d,N,p) = \frac{1}{N!}\prod_{i=1}^{N}\oint \frac{dz_i}{2\pi i}\frac{1}{z_i^{p+1}}(z_i+1)^{d-2}\prod_{k<l}|z_k - z_l|^2 \tag{5.9}$$

where the integration contour is the unit circle around the origin in $\mathbb{C}$, $\forall i$. Now, we perform a new change of variables, passing from the unit complex circle to the real line by means of stereographic projection: $z_j = \frac{i-x_j}{i+x_j}$. The integral becomes

$$Dof(d,N,p) = \frac{2^{N^2+Nd-3N}}{N!\pi^N}\int_{\mathbb{R}^N} d^N x\, |\Delta(x)|^2 \prod_{j=1}^{N}(1+ix_j)^{-(N+p)}(1-ix_j)^{-(d+N-p-2)} \tag{5.10}$$

where we have recognized the square of the Van der Monde determinant

$$\Delta(x) = \prod_{i<j}(x_i - x_j)\,. \tag{5.11}$$

Written in term of the x_i variables, (5.10) is seen to belong to a wide class of Selberg's integrals, that can be computed by means of orthogonal polynomials techniques.[4] The known Selberg's integral in question, that can be found in [18], reads

$$\begin{aligned} J(a,b,\alpha,\beta,\gamma,n) &= \int_{\mathbb{R}^N} d^N x\, |\Delta(x)|^{2\gamma}\prod_{j=1}^{n}(a+ix_j)^{-\alpha}(b-ix_j)^{-\beta} \\ &= \frac{(2\pi)^n}{(a+b)^{(\alpha+\beta)n-\gamma n(n-1)-n}}\prod_{j=0}^{n-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta-(n+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)} \end{aligned} \tag{5.12}$$

valid for $\mathrm{Re}a$, $\mathrm{Re}b$, $\mathrm{Re}\alpha$, $\mathrm{Re}\beta > 0$, $\mathrm{Re}(\alpha+\beta) > 1$, and

$$-\frac{1}{n} < \mathrm{Re}\gamma < \min\left(\frac{\mathrm{Re}\alpha}{n-1}, \frac{\mathrm{Re}\beta}{n-1}, \frac{\mathrm{Re}(\alpha+\beta+1)}{2(n-1)}\right)\,.$$

Our eq. (5.10) corresponds to this form of the Selberg's integral with ($a = b = \gamma = 1$, $\alpha = N + p$, $\beta = d + N - p - 2$, $n = N$) so, with (5.12) at hand, after a little algebra, we obtain the final result

$$\begin{aligned} Dof(d,N,p) &= \frac{2^{N^2+Nd-3N}}{N!\pi^N} J(1,1,N+p,d+N-p-2,1,N) \\ &= \prod_{j=0}^{N-1}\frac{j!(j+d-2)!}{(j+p)!(j+d-2-p)!} \end{aligned} \tag{5.13}$$

that agrees with the dimension of the rectangular Young tableau of $\mathrm{U}(d-2)$ with p rows and N columns, as in (3.32), thus reproducing the number of physical polarizations predicted by canonical quantization.

[4]Much information and many details about these techniques can be found in [18].

6 Conclusions and outlook

We have analyzed U(N) spinning particles and obtained from them new gauge invariant higher spin equations that live on complex spaces. These equations define a complex version of the standard higher spin equations of Minkowski spacetime [8, 12, 13]. We have obtained them by integrating a subset of the constraints that arise form the Dirac quantization of the U(N) spinning particle. The spinning particle language is quite efficient, as already exemplified in [5] for the $O(N)$ spinning particle, in which case it allowed to describe in a simple way the structure of minkowskian higher spin fields, including the use of compensators [9] and the application of generalized Poincaré lemmas to integrate higher order field equations [19–22]. Similar constructions have been presented here for the new class of complex higher spin equations. Having described these equation on a flat complex manifold, we have shown in principle their consistency also on a more general class of Kähler manifolds, namely those Kähler manifolds with constant holomorphic curvature, as in this case the algebra of the quantum constraints closes in a quadratic way and remains first class. An important feature of this algebra is that it realizes in a geometrical way the zero mode sector of the nonlinear two dimensional U(N) superconformal algebras, introduced sometimes ago by Bershadsky and Knizhnik [10, 11]. Finally we have considered the path integral quantization on the circle of the U(N) spinning particle in flat space, corresponding to the one-loop effective action of the quantized version of the higher spin equations introduced earlier. This way we have calculated the number of physical degrees of freedom for all d, N and p, and checked the correctness of our path integral construction containing, in particular, the measure on the moduli space of the U(N) extended supergravity on the circle.

As for future developments, an application of this worldline approach could be to compute perturbatively the one-loop effective action on arbitrary Kähler manifolds for the U(1) and U(2) models, as done for the similar cases of the $O(N)$ spinning particle on arbitrarily curved spaces with $N = 0, 1, 2$, which produced the effective action for scalars [23], spin 1/2 [24], and arbitrary differential forms (including vectors) [25] coupled to gravity, respectively. Similarly, one could consider the U(N) models with $N > 2$ on Kähler manifolds with constant holomorphic curvature and compute the corresponding partition function on the circle (i.e. the one loop effective action of the corresponding quantum field theory). Finally, it could be interesting to study along similar lines gauged versions of various quantum mechanical models, for example those described in [26–29], to unearth novel gauge invariant field equations.

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